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Forced vibrations of an elastic circular plate supported by unilateral edge lateral springs

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Abstract. The present study deals with forced vibrations of an elastic circular plate supported along its circular edge by unilateral elastic springs. The plate is assumed to be subjected to a uniformly distributed and a concentrated load. Under the combination of these loads, equations of motion are explicitly derived for static and dynamic response analyses by assuming a series of the displacement functions of time and other unknown parameters which are to be determined by employing Lagrangian functional. The approximate solution is sought by applying the Lagrange equations of motions by using the potential energy of the external forces that includes the contributions of the edge forces and the external moments, i.e., those of the effects of the boundary condition to the analysis. For the numerical treatment of the problem in the time domain, the linear acceleration procedure is adopted. The tensionless character of the support is taken into account by using an iterative process and, the coordinate functions for the displacement field are selected to partially fulfill the boundary conditions so that an acceptable approximation can be achieved faster. Numerical results are presented in the figures focusing on the nonlinearity of the problem due to the plate lift-off from the unilateral springs at the edge support.

Keywords: elastic circular plate; forced vibrations; static and dynamic response; tensionless support

1. Introduction

Response of elastic plates on an elastic foundation is of theoretical and practical interest in the structural, geotechnical, and mechanical engineering problems, such as, in aeronautics, spacecraft, automobiles, bridges, etc. There are theoretical and numerical studies on this subject in the open literature. In these studies, several different foundation models with various degrees of sophistication have been assumed depending on the complexity of the soil behavior to be taken into account. In fact, soil exhibits inherently a very complex behavior; the simplest being the Winkler model which assumes the soil to be consisting of separate elastic springs. In the conventional Winkler foundation model, it is assumed that the soil reaction at any point is proportional to the displacement of the soil at that same point. The response of structural elements resting on the conventional Winkler foundation is analyzed assuming that the foundation supports compressive as well as tensile stresses, which simplifies the analysis considerably. However, this assumption is questionable or not valid for a base soil that supports compressive reactions only. The difficulty of the problems involving compression supports only; is that the contact region of the base is not known in advance and varies with load. When separation at the supports occurs, the analysis becomes nonlinear and its numerical treatment requires an iterative procedure in general; because the soil does not support any tensile reactions. Consequently, only limited studies dealing with the unilateral, i.e., tensionless Winkler foundations have been published. For the cases in which the loads are of dynamic character and inertia forces are included, the number and the extent of the studies published are very limited. For the problems involving circular and cartesian symmetry, the solutions are often obtained for circular and rectangular plates by adopting iterative numerical techniques to the nonlinear governing equations of the problem. Analytical aspects of the structures supported on various foundation models have been discussed by Kerr (1964, 1976, 1996), where it has been pointed out that for the finite beam that rests on a Winkler foundation, the intensity of the load does affect the location of the point of separation of beam and base. Furthermore, it has been indicated that intuitive formulation of the boundary conditions may lead to incorrect boundary conditions and special care should be taken, especially when a two-parameter foundation, such as the Wieghardt foundation, is considered. The lift-off problems are of practical interest; when the perimeter of the plate is completely free, or it is unilaterally simply supported so that it can lift-off from the support. There are various papers dealing with the static behavior of plates on a unilateral Winkler foundation. In the lift-off problems, it often attracts interest to examine the extent of the lift-off region from the non-tension foundation. Weitsman (1970) studied the problem of a laterally loaded plate on an elastic foundation subjected to a concentrated load by considering an infinitely extended

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plate on an elastic foundation that reacts in compression only, by paying particular attention to the separation of the plate from the foundation. Kamiya (1977) studied the axially symmetric behavior of a circular plate clamped along its edge on a bi-modulus and no-tension foundation. Equilibrium configuration of a rectangular plate, supported on a unilateral elastic Winkler foundation that reacts in compression only, is investigated by Villaggio (1983) by assuming the plate to be weightless and subjected to a concentrated load at its midpoint. The contact curve where the plate loses contact with the foundation is studied. Dempsey *et al.* (1984) investigated separation of corners of a simply supported rectangular plate on a no-tension foundation and determined the extent of lift-off, where the global equilibrium of the vertical forces is checked by evaluating the support reactions. A circular plate resting on a unilaterally elastic support around its periphery subjected to a uniformly distributed load along a circular arc is considered and analysis is accomplished by dividing the plate into two parts by a cylindrical section and establishing the continuity conditions along their boundary in addition to the edge conditions (Celep *et al.* 1988a). Investigation of the completely free circular plate supported on an elastic Winkler foundation is carried out by Celep (1988), where the plate is assumed to be under an eccentric vertical load. The analysis is carried out by minimizing the total potential energy of the system. Contact between a circular plate and its no-tension edge support is studied by employing the free vibration mode shapes of the completely free plate by satisfying the lift-off boundary condition approximately harmonic average by Celep *et al.* (1988b). Hong *et al.* (1999) addressed the axially symmetric shells and plates on a tensionless elastic foundation by employing a finite element method. The analysis is validated by comparison with existing results for circular plates and beams on tensionless foundations. A numerical methodology for analysis of plates resting on tensionless elastic foundations was presented by Silva *et al.* (2001) by using the finite element method. The significance of the methodology is illustrated by comparative examples. Some of these static problems are generalized to include the dynamic loads and oscillations of the plate. Analysis of axisymmetric vibrations of a circular plate subjected to a concentrated vertical dynamic load at its center is carried out by Celep and Turhan (1990) by assuming the plate on a no-tension foundation. The solution is accomplished by applying Galerkin's method and by using the modal functions of the completely free plate. The responses of a plate-column system on a tensionless Winkler foundation subjected to free vibration, harmonic ground motion, and to El Centro 1940 earthquake are investigated by focusing on the tensionless character of the foundation (Celep 1992). Güler and Celep (1995) studied static and dynamic responses of a thin circular plate on an elastic foundation of Winkler type that reacts in compression-only by applying Galerkin's method. The plate is assumed to be subjected to time-dependent external moment and vertical load at its center. The governing equations of the problem are reduced to a set of coupled differential equations of second order having time-dependent coefficients.

Static and dynamic behavior of a rigid circular plate supported by a tensionless Winkler support along the edge of the plate and that of a circular elastic plate on a tensionless foundation are investigated by Celep *et al.* (2003, 2004), respectively. In the second study, the load and the geometry of the problem is assumed to be rotationally symmetric. The analysis is accomplished by applying Galerkin method to the governing equations of the problem. Attention is paid to the contact condition and to the circumferentially distributed edge forces which merge, when the complete contact is established. Studies related to the response of a beam and a plate and rectangular plate-column system on a tensionless Pasternak foundation subjected to static and dynamic loads are accomplished by Celep *et al.* (2011, 2005, 2007). Static response of a linear elastic Euler-Bernoulli beam on a tensionless foundation subjected to vertical load is investigated by Zhang and Murph (2004). The solution is obtained without employing any assumption about the contact area and it is pointed out that the contact area is a sensitive function of the beam length and may depend on the magnitude of the load. In a recent review on the interaction between structural elements and supporting foundations Wang *et al.* (2005) highlighted the key areas, including the foundation modeling, and analytical and numerical methods used in the analysis of the interaction between them. Attar *et al.* (2016) studied static and dynamic analyses of Euler-Bernoulli beams on a viscoelastic foundation with a unilateral contact constraint. The solution is accomplished by considering the non-smooth continuous system to a piecewise smooth multi-degree of freedom model, i.e., by employing a chain of discrete units and using the lattice spring model to handle the unilateral constraints at the interface. Robustness of the proposed solution method dealing with unilateral constraints is demonstrated in various examples. It is pointed out that the contact length of the beam to the foundation depends on load in the presence of the gap, whereas it is independent of the applied load; when the gap is zero. A beam on a tensionless Winkler elastic foundation is considered for the derivation of the conditions of complete contact between the beam and the foundation by employing Mathematica and the Galerkin method by Ioakimidis (2016). Generalization of the solution method to the two-parameter elastic foundations is briefly discussed as well. Zhang *et al.* (2018) drew attention to the qualitative difference between the responses of a beam; on the conventional, the tensionless, and the bilinear elastic Winkler foundations. Considering the conventional and the tensionless Winkler foundations to be the two special cases of the bilinear elastic foundation model, it is pointed out that the numerical method developed for the bilinear foundation can be applied to the other two models as well regardless of the extent of the contact area. Infinitely beams under axial and lateral loads on a tensionless Winkler foundation are investigated by Zhang *et al.* (2019) by focusing on the closed-form solution. The deflection of a beam subjected to various axial and transverse concentrated loads is studied. Since the contact zone of a tensionless contact is not known a priori, the corresponding mathematical difficulties are discussed, and five closed-form solutions are derived for use as a guideline for further study. Bhattiprolu *et al.* (2013, 2014, 2016) considered a pinned-pinned beam on a conventional and a tensionless Winkler foundation and obtained the response to static as well as dynamic behavior by employing Galerkin's method and the modes of the corresponding linear problem. In the tensionless foundation case, variation of the contact region and the lift-off points with beam motion is considered. Capability of the solution method for the complicated loading cases is demonstrated in various example numerical solutions. An analytical model for a buried beam on a tensionless foundation subjected to differential settlement is examined by Zhang *et al.* (2020) by dividing the beam into three segments: two semi-infinite foundation beams and a middle finite beam separated from the ground. Later, the equations of the beams are combined with the continuity conditions. Numerical results are verified with those of the methods used in the existing literature.

Advanced plate and shell theories are considered in various studies for similar analyzes of thick plates recently. Lezgy-Nazargah and Cheraghi (2015) presented a three-dimensional solution for the bending analysis of functionally graded and layered neutral magneto-electro-elastic plates resting on two-parameter elastic foundations by considering imperfect interfacial bonding. In the equations of motion consisting of Gauss' equations for electrostatics and magnetostatics, the boundary and interface conditions, are satisfied exactly. The state-space method is employed for solving the governing partial differential equations. The exact solution is obtained by considering the effects of two-parameter elastic foundation, gradient index, bonding imperfection, and applied mechanical and electrical loads on the response of the functionally graded magneto-electro-elastic plate. Lezgy-Nazargah (2016) developed a methodology for bending and vibration analysis of thick plates by proposing mixed variational formulation by combining the concept of Reissner's mixed variational theorem and Rong's generalized mixed variational theorem. The corresponding numerical results show that the proposed mixed plate theory is capable of achieving the accuracy of nearly the exact three-dimensional theory of elasticity. Lezgy-Nazargah and Meshkani (2018) developed a four-node quadrilateral partial mixed plate element static and free vibration analysis of functionally graded plates resting on Winkler-Pasternak elastic foundations. The finite element model they present considers the effects of shear deformations and normal flexibility of the functionally graded material plates without using any shear correction factor. It also fulfills the boundary conditions of the transverse shear and the normal stresses on the top and bottom surfaces of the plate. Besides these capabilities, the number of unknown field variables of the plate is only six. The numerical results are verified with those of the three-dimensional theory of elasticity and those of the classical and high-order plate theories. Lezg-Nazargah *et al.* (2022) developed a mixed-field plate finite element model for the analysis of shallow footings resting on soil foundations. The tensionless nature of the supporting soil foundation is considered by adopting an incremental, iterative procedure. The continuity of displacements at the interface between the shallow footing and soil is fulfilled using the penalty approach. Results of the method they present show that the method is an efficient and accurate tool for solving the problems of shallow footings resting on subsoil.

Various studies dealing with advanced shell theories can be found as well. Liu *et al.* (2021) studied nonlinear dynamic responses of sandwich functionally graded porous cylindrical shells consisting of three layers and embedded in elastic media. The governing equation of the problem is derived by using the improved Donnell's nonlinear shell theory and Hamilton's principle. Later, the Galerkin method is used to transform the governing equations into nonlinear ordinary differential equations, and an approximate analytical solution is obtained by using the multiple scales method. Numerical studies are presented on the effects of various system parameters. Liu *et al.* (2021) studied nonlinear forced vibrations of functionally graded material sandwich cylindrical shells with porosities on an elastic substrate. An energy approach is employed to obtain the nonlinear equations of motion by using Donnell's nonlinear shallow shell theory and Hamilton's principle. Effects of the core-to-thickness ratio, porosity volume fraction, power-law exponent, and external excitation on nonlinear forced vibration characteristics of sandwich shells with porosities are investigated. Liu *et al.* (2021) developed an approach for the solution of nonlinear forced vibrations of functionally graded piezoelectric shells subjected to electric-thermo-mechanical loads by considering micro-voids. Equations of motions are obtained by using Hamilton's principle and the Donnell nonlinear shallow shell theory by combining the multi-mode Galerkin scheme and the pseudo-arclength continuation method. The results show that the externally applied voltage, temperature change, external excitation, power-law exponent, and porosity volume fraction play important roles in nonlinear vibration response and bifurcation analysis of the shells considered. Liu *et al.* (2022) investigated the nonlinear forced vibrations in multi-physics fields, a coupled nonlinear modeling for composite cylindrical shells by using the improved Donnell nonlinear shell theory and Maxwell static electricity/magnetism equations. The nonlinear governing equations of the problem are obtained by using Hamilton principle and Galerkin technique. Numerical results show that the present procedure can solve coupled multi-modes nonlinear partial differential equations.

In the case of dynamic problems, i.e., for free or forced vibrations of the plate on tensionless supports, the contact region between the plate and the supports varies with time. The numerical solution is implemented by employing stepwise integration along the time domain by updating the contact region continuously. The present study is carried out by considering a circular elastic plate supported along its edge by unilateral elastic springs, i.e., no-tension edge support. The plate is assumed to be subjected to a uniformly distributed load and a concentrated load. Because the problem deals with the lift-off, the governing equations have an inherently high degree of nonlinearity. In the solution, the coordinate functions for the displacement field

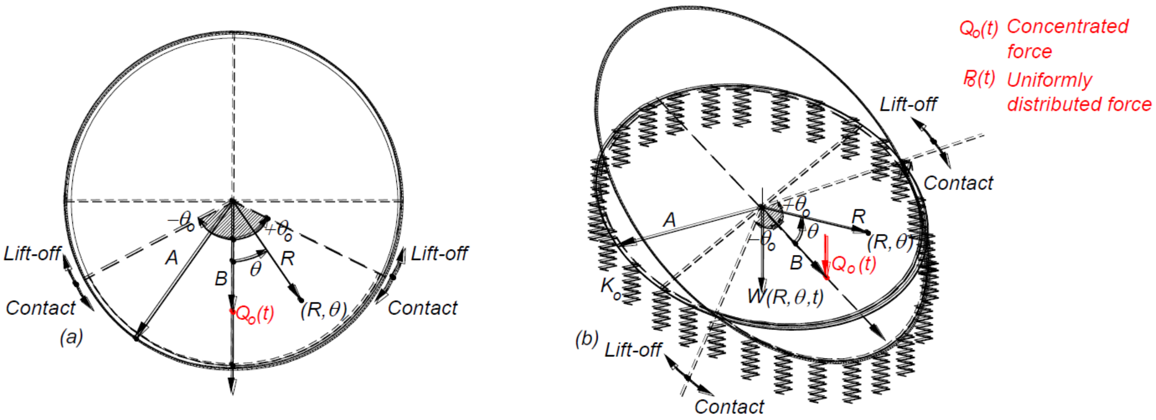

Fig. 1 Circular plate subjected to a uniformly distributed load $P_o(t)$ and a concentrated force $Q_o(t)$ on unilateral edge elastic support with elastic spring constant K_o

are selected to partially fulfill the boundary conditions so that an acceptable approximation can be achieved faster. Static and the dynamic responses of the plate are studied, and the numerical results are illustrated in the figures for comparison. An approximate solution is carried out by using the total potential energy and applying the Lagrangian functional. The approximate solution is obtained by applying the Lagrange equations of motions by using the potential energy of the external forces that includes the contributions of the edge forces and the external moments, i.e., those of the effects of the boundary condition to the analysis. Special attention is paid to the boundary conditions and the global force equilibrium in the static and dynamic cases, by including the inertia forces.

2. Statement of the problem and equation of motion

Consider a completely free circular elastic thin plate of radius A, bending stiffness $D = Eh^3/12(1-\nu^2)$, modulus of elasticity E, Possion's ratio ν and mass per unit area m. The plate is supported along its edge with unilateral elastic springs having a stiffness K, as Fig. 1 shows. Motivated from the practical application, the circular plate is assumed to be subjected to a concentrated load $Q_o(t)$ applied with an eccentricity of B from the center of the plate and a uniformly distributed load $P_o(t)$. Since the geometry and the loading of the problem are symmetric with respect to the axis of the plate, the vertical displacement and the support reactions of the plate are expected to have the same symmetry. The loads are assumed to be time-dependent, consequently, the parameters of the system depend on the radius R, the angle θ and on the time t. Since the unilateral support reacts in compression only, a possible lift-off of the plate from the support is expected. The contact and the lift-off regions are separated by the contact angle θ_o, i.e., the region of the support $\theta_o \geq \theta \geq 0$ corresponds to the contact region and the region of the support $\pi \geq \theta \geq \theta_o$ corresponds to the lift-off region along the edge of the plate. In this case, it is difficult to find the exact solution for the displacement function $W(R,\theta,t)$, when it is not impossible, due to various facts, such as the eccentricity of concentrated load. Time dependency of the problem and the initial conditions of the dynamic problem create another challenge. Assumption of unilateral supports leads to the problem that the contact angle θ_o is not known in advance, and it depends on the parameters of the system as well as on the loading. Furthermore, it should be remembered that it depends on time in the case of dynamic loading.

The approximate solution is sought by applying the Lagrange equations of motions, i.e.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad i = 1,2,3,... \tag{1}$$

where the Lagrangian is expressed as $L=V\text{-}T$, where T is the kinetic energy. The potential energy is written as $V=U\text{-}W_o$, where U and W_o are the strain energy of the elastic plate and the work done by the external forces, respectively. They are written as follows for the elastic circular plate considered in the present paper

$$\begin{aligned} T &= \frac{1}{2}\int_0^a\int_0^{2\pi} m\frac{\partial^2 W(R,\theta,t)}{\partial t^2} R\,dR\,d\theta \\ U &= \frac{D}{2}\int_0^a\int_0^{2\pi}\left[\left(\frac{\partial^2 W}{\partial R^2} + \frac{1}{R}\frac{\partial W}{\partial R} + \frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2}\right)^2\right. \\ &\quad -2(1-\nu)\left[\frac{\partial^2 W}{\partial R^2}\left(\frac{1}{R}\frac{\partial W}{\partial R} + \frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right] \\ &\quad \left.-\left(\frac{1}{R}\frac{\partial^2 W}{\partial R\partial\theta} - \frac{1}{R^2}\frac{\partial W}{\partial \theta}\right)^2\right] R\,dR\,d\theta \end{aligned} \tag{2}$$

In the present problem, the plate is assumed to be supported by unilateral springs only, the boundary conditions of the plate can be expressed as

$$\begin{aligned} V_r(R=A,\theta,t) &= K\,H(\theta,t)W(R=A,\theta,t) \\ M_r(R=A,\theta,t) &= 0 \end{aligned} \tag{3}$$

where

$$V_r = -D\left[\frac{\partial}{\partial R}\left(\frac{\partial^2 W}{\partial R^2}+\frac{1}{R}\frac{\partial W}{\partial R}+\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right.$$
$$\left.+(1-\nu)\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial W}{\partial R}-\frac{1}{R}W\right)\right] \qquad (4)$$
$$M_r = -D\left[\frac{\partial^2 W}{\partial R^2}+\frac{\nu}{R}\left(\frac{\partial W}{\partial R}+\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}\right)\right]$$

The contact function $H(\theta,t)$ is introduced in the boundary conditions to include the tensionless character of the support and it is defined as

$$H(\theta,t)=1 \text{ for } W(R=A,\theta,t)\geq 0\,,$$
$$H(\theta,t)=0 \text{ for } W(R=A,\theta,t)<0\,. \qquad (5)$$

For the present problem, the potential energy of the external forces can be expressed as follows

$$W_o = P_o\int_0^a\int_0^{2\pi} W(R,\theta,t)\,R\,dR\,d\theta + Q_o\,W(B,0,t)$$
$$-\int_0^{2\pi}\left(V_r-\frac{1}{2}K\,H(\theta,t)W(A,\theta,t)\right)W(A,\theta,t)A\,d\theta \qquad (6)$$
$$-\int_0^{2\pi} M_r\frac{\partial W(A,\theta,t)}{\partial R}A\,d\theta$$

where the first-two terms correspond to the work done by the uniform load P_o and the concentrated load Q_o, the third and fourth terms correspond to the contribution of the edge forces and the external moments to the potential energy. The displacement function of the plate will be approximated by using the mode shapes of the completely free circular plate. The difference between the free-end boundary conditions of the completely free plate and the unilateral edge of the plate is compensated in the third term by adding the effect of the shear force and the support force of the plate. The property of the unilateral spring is taken into account by including the contact function $H(\theta,t)$. For the approximate solution, the coordinate functions for the displacement $W(R,\theta,t)$ are used. The expression (6) can be simplified, when the selected coordinate functions satisfy the boundary conditions, as it will be discussed below.

The displacement function of the plate $W(R,\theta,t)$ is approximated as

$$W(R,\theta,t)= A\,w(r,\theta,t)=$$
$$A\left[\bar{T}_o(t)+\bar{T}_1(t)\,r\cos\theta+\sum_{n=0}^{\infty}T_n(t)\,w_n(r)\cos(n\theta)\right], \qquad (7)$$

where $r = R/A$ and

$$w_n(r) = J_n(\lambda_n r)+A_n I_n(\lambda_n r) \qquad (8)$$

where J_n and I_n are the regular and the modified Bessel functions of the first kind (McLachlan 1955). The coordinate functions are selected to reflect symmetry of the present problem. Furthermore, since the plate is allowed to move vertically and rotate along its axis, a rigid translation and a rotation are included into the coordinate functions which are the first and the second terms in Eq. (7), where $\bar{T}_o(t)$, $\bar{T}_1(t)$ and $T_n(t)$ $n = 0,1,2,3,\ldots$ are time dependent parameters. The other two unknown parameters λ_n and A_n are to be determined to satisfy the boundary conditions. However, the boundary conditions are difficult to satisfy completely due to their complexity, i.e., in case of the presence of the lift-off and the contact region along the support. Adopting an approximate approach, the coordinate functions are assumed to be the free vibration mode shapes of the completely free circular plate, i.e., being free from the bending moment and the shearing force at the edge of the plate

$$M_r(R=A,\theta)=0 \qquad V_r(R=A,\theta)=0 \qquad (9)$$

which are expressed as

$$\frac{d^2 w_n}{dr^2}+\nu\left(\frac{dw_n}{dr}-n^2 w_n\right)=0$$
$$\frac{d}{dr}\left(\frac{d^2 w_n}{dr^2}+\frac{1}{r}\frac{dw_n}{dr}-n^2\frac{w_n}{r^2}\right)-n^2(1-\nu)\left(\frac{dw_n}{dr}-w_n\right)=0$$
$$\text{for } r=1 \qquad (10)$$

which yield

$$\left[n(n-1)(1-\nu)-\lambda_n^2\right]J_n(\lambda_n)+\lambda_n(1-\nu)J_{n+1}(\lambda_n)+$$
$$A_n\left\{\left[n(n-1)(1-\nu)+\lambda_n^2\right]I_n(\lambda_n)-\lambda_n(1-\nu)I_{n+1}(\lambda_n)\right\}=0$$
$$\left[n(n-1)(1-\nu)+\lambda_n^2\right]J_n(\lambda_n)-\lambda_n\left[\lambda_n^2+n^2(1-\nu)\right]J_{n+1}(\lambda_n)+$$
$$+A_n\left\{\left[n(n-1)(1-\nu)-\lambda_n^2\right]I_n(\lambda_n)\right.$$
$$\left.-\lambda_n\left[\lambda_n^2-n^2(1-\nu)\right]I_{n+1}(\lambda_n)\right\}=0 \qquad (11)$$

where ν denotes Poisson's ratio. This system has infinite number of roots for A_n and λ_n, which control mode shapes in the radial direction and the shape in the angular direction. However, for the approximate solution only limited number of roots is assumed to be satisfactory.

Since the mode shapes of the completely free plate satisfy the boundary conditions (10) the potential energy of the external loads can be written as

$$\frac{W_o}{D}=-p_o\int_0^1\int_0^{2\pi} w(r,\theta,t)\,r\,dr\,d\theta - q_o\,w(b,0,t)$$
$$+k\int_0^{\theta_o} H(\theta,t)\,w^2(1,\theta,t)\,d\theta \qquad (12)$$

The edge $-\theta_o \leq \theta \leq \theta_o$ corresponds to the edge of the plate, where the contact between the plate and support is established. The non-dimensional parameters of the problem are defined as

$$q_o=\frac{Q_o A}{D} \quad p_o=\frac{P_o A^3}{D} \quad k_o=\frac{K_o A^3}{D} \quad \tau = t\sqrt{\frac{D}{mA^4}} \qquad (13)$$

Substituting the displacement function $W(R,\theta,t)$ (12) into the Lagrangian $L(\bar{T}_o,\bar{T}_1,\dot{\bar{T}}_o,\dot{\bar{T}}_1,T_n,\dot{T}_n)$ (1) leads to the following relation

$$
\begin{aligned}
L = {} & \pi D\left\{ T_o^2 \lambda_o^4 \int_0^1 \left[-J_o(\lambda_o r)+A_o I_o(\lambda_o r)\right]^2 r dr \right.\\
& \left. +\frac{1}{2}\sum_{n=1}^{\infty}\left[T_n^2 \lambda_n^4 \int_0^1 \left[-J_n(\lambda_n r)+A_n I_n(\lambda_n r)\right]^2 r dr \right]\right\}\\
& -\pi D(1-\nu)\left[2T_o^2 \int_0^1 \frac{dw_o}{dr}\frac{d^2 w_o}{dr^2} dr + \sum_{n=1}^{\infty}\left[T_n^2 \int_0^1 \frac{dw_n}{dr}\frac{d^2 w_n}{dr^2} dr \right]\right.\\
& \left. -\sum_{n=1}^{\infty} n^2 \left[T_n^2 \int_0^1 w_n \frac{d^2 w_n}{dr^2}\frac{dr}{r}\right]\right]\\
& -\pi D(1-\nu)\left[-\sum_{n=1}^{\infty} n^2 \left[T_n^2 \int_0^1 \frac{dw_n}{dr}\frac{dw_n}{dr}\frac{dr}{r}\right]\right. \cdot\\
& \left. -\sum_{n=1}^{\infty} n^2 \left[T_n^2 \int_0^1 w_n w_n \frac{dr}{r^3}\right] + 2\sum_{n=1}^{\infty} n^2 \left[T_n^2 \int_0^1 w_n \frac{dw_n}{dr}\frac{dr}{r^2}\right]\right]\\
& + D k_o \theta_o \bar{T}_o^2 + 0.5 D k_o (\theta_o + 0.5\sin 2\theta_o)\bar{T}_1^2\\
& + D k_o \theta_o w_o^2(1) T_o^2 + 2D k_o w_o(1) w_1(1) T_o T_1 \sin\theta_o\\
& + D k_o \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\left(\frac{\sin(n-m)\theta_o}{n-m}+\frac{\sin(n+m)\theta_o}{n+m}\right)\right.\\
& \left. w_n(1) w_m(1) T_n T_m \right] + 2D k_o \bar{T}_o \bar{T}_1 \sin\theta_o\\
& + 2D k_o w_o(1)\theta_o \bar{T}_o T_o + 2D k_o \bar{T}_o \sum_{n=1}^{\infty}\left[\frac{\sin(n\theta_o)}{n} w_n(1) T_n\right]\\
& + 2D k_o w_o(1) \bar{T}_1 T_o \sin\theta_o\\
& + D k_o \bar{T}_1 \sum_{n=1}^{\infty}\left[\left(\frac{\sin(n-1)\theta_o}{n-1}+\frac{\sin(n+1)\theta_o}{n+1}\right) w_n(1) T_n\right]\\
& + D\pi\left\{ 0.5\dot{\bar{T}}_o^2 + 0.125\dot{\bar{T}}_1^2 + \dot{T}_o^2 \int_0^1 w_o^2 r dr \right.\\
& \left. + \sum_{n=1}^{\infty}\left[0.5\dot{T}_n^2 \int_0^1 w_n^2 r dr\right] + 2\dot{\bar{T}}_o \dot{T}_o \int_0^1 w_o r dr + \dot{\bar{T}}_1 \dot{T}_1 \int_0^1 w_1 r^2 dr \right\}\\
& - q_o\left[\bar{T}_o + \bar{T}_1 b + 2\sum_{n=1}^{\infty} T_n w_n(b)\right] - \bar{T}_o p_o \pi
\end{aligned}
\tag{14}
$$

where the time derivatives are with respect to the non-dimensional time τ. Lagrange equations of motion (1) can be obtained by setting the partial derivatives of the Lagrangian functional with respect to $\bar{T}_o$, $\bar{T}_1$ and T_n. Thus, the following nonlinear system of differential equations of second order is obtained

$$\mathbf{M}\ddot{\mathbf{T}}(\tau)+\mathbf{K}(\tau)\mathbf{T}(\tau)=\mathbf{P}(\tau) \tag{15}$$

where $\mathbf{M}$ is the mass symmetric matrix and $\boldsymbol{K}(\tau)$ is the nonlinear stiffness symmetric matrix which depends on the displacement configuration. Furthermore, $\boldsymbol{P}(\tau)$ is the time dependent loading vector and $\boldsymbol{T}(\tau)$ the displacement vector. The non-zero elements of the mass matrix and the loading vector are defined as follows

$$\mathbf{T}^T(\tau) = \left[\bar{T}_o, \bar{T}_1, T_o, T_1, T_2, \ldots\right]$$

$$M_{11} = \pi \quad M_{13} = 2\pi\int_0^1 w_o r dr \quad P_1 = \pi p_o + q_o$$

$$M_{22} = \pi/4 \quad M_{24} = \pi\int_0^1 w_1 r^2 dr \quad P_2 = q_o b$$

$$M_{33} = 2\pi\int_0^1 w_o^2 r dr \quad M_{44} = \pi\int_0^1 w_1^2 r dr \quad P_3 = q_o w_o(b)$$

$$M_{nn} = \pi\int_0^1 w_n^2 r dr \quad P_4 = q_o w_1(b) \quad P_{n+3} = q_o w_n(b)$$

$$K_{11} = 2k_o\theta_o \quad K_{12} = 2k_o\sin\theta_o \quad K_{13} = 2k_o w_o(1)\theta_o$$

$$K_{14} = 2k_o w_1(1)\sin\theta_o \quad K_{1,n+3} = 2k_o w_n(1)\frac{\sin\theta_o}{n}$$

$$K_{22} = k_o(\theta_o + 0.5\sin 2\theta_o) \quad K_{23} = 2k_o w_o(1)\sin\theta_o$$

$$K_{24} = k_o w_1(1)(\theta_o + 0.5\sin 2\theta_o)$$

$$K_{2,n+3} = k_o w_n(1)\left[\frac{\sin(n-1)\theta_o}{n-1}+\frac{\sin(n+1)\theta_o}{n+1}\right]$$

$$
\begin{aligned}
K_{33} = {} & 2\pi\lambda_o^4\int_0^1\left[-J_o(\lambda_o r)+A_o I_o(\lambda_o r)\right]^2 r dr\\
& -4\pi(1-\nu)\int_0^1 \frac{dw_o}{dr}\frac{d^2 w_o}{dr^2} dr + 2k_o\theta_o w_o^2(1)
\end{aligned}
$$

$$K_{34} = 2k_o w_o(1) w_1(1)\sin\theta_o$$

$$
\begin{aligned}
K_{44} = {} & \pi\lambda_1^4\int_0^1\left[-J_1(\lambda_1 r)+A_1 I_1(\lambda_1 r)\right]^2 r dr\\
& -2\pi(1-\nu)\int_0^1 \frac{dw_1}{dr}\frac{d^2 w_1}{dr^2} dr + 2\pi(1-\nu)\int_0^1 w_1 \frac{d^2 w_1}{dr^2}\frac{dr}{r}\\
& +2\pi(1-\nu)\int_0^1\left(\frac{d^2 w_1}{dr^2}\right)^2\frac{dr}{r} + 2\pi(1-\nu)\int_0^1 (w_1)^2\frac{dr}{r^3}\\
& -4\pi(1-\nu)\int_0^1 w_1\frac{dw_1}{dr}\frac{dr}{r^2} + k_o w_1^2(0.5\sin 2\theta_o + \theta_o)
\end{aligned}
$$

$$K_{4,n+3} = k_o w_1(1) w_n(1)\left[\frac{\sin(n-1)\theta_o}{n-1}+\frac{\sin(n+1)\theta_o}{n+1}\right]$$

$$
\begin{aligned}
K_{n+3,n+3} = {} & \pi\lambda_n^4\int_0^1\left[-J_n(\lambda_n r)+A_n I_n(\lambda_n r)\right]^2 r dr\\
& -2\pi(1-\nu)\int_0^1 \frac{dw_n}{dr}\frac{d^2 w_n}{dr^2} dr + 2\pi n^2(1-\nu)\int_0^1 w_n \frac{d^2 w_n}{dr^2}\frac{dr}{r}\\
& +2\pi n^2(1-\nu)\int_0^1\left(\frac{d^2 w_n}{dr^2}\right)^2\frac{dr}{r} + 2\pi n^2(1-\nu)\int_0^1 (w_n)^2\frac{dr}{r^3}\\
& -4\pi n^2(1-\nu)\int_0^1 w_n\frac{dw_n}{dr}\frac{dr}{r^2} + k_o w_n^2\left(\theta_o + \frac{\sin 2n\theta_o}{2n}\right)
\end{aligned}
$$

$$K_{n+3,m+3} = k_o w_n(1) w_m(1)\left[\frac{\sin(n-m)\theta_o}{n-m}+\frac{\sin(n+m)\theta_o}{n+m}\right]$$

$$n \neq m$$

$$n = 2,3,4,\ldots \qquad m = 2,3,4,\ldots \tag{16}$$

Global vertical force equilibrium in the plate by including the inertia forces, the external loads and the support reaction at the edge of the plate can be expressed as follows

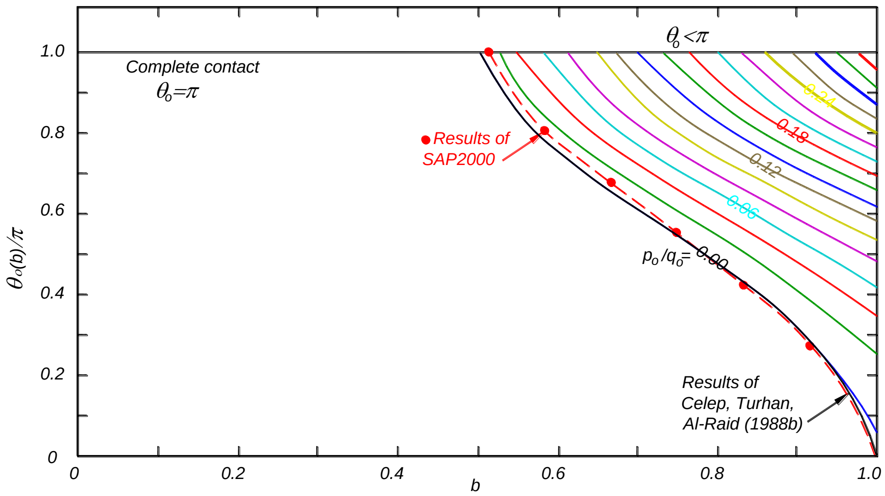

Fig. 2 Contact angle $\theta_o(b)$ for various values of the loading ratio p_o/q_o

$$P_o\pi A^2 + Q_o = \int_0^{2\pi} H(\theta,t)KW(A,\theta,t)A\,d\theta + \int_0^A\int_0^{2\pi} m\ddot{W}(R,\theta,t)RdRd\theta \tag{17}$$

This equilibrium can be expressed by using the nondimensional parameter as defined above as

$$p_o\pi + q_o = k_o\int_{-\theta_o}^{\theta_o} w(1,\theta,\tau)\,d\theta + \int_0^1\int_0^{2\pi}\ddot{w}(r,\theta,\tau)rdrd\theta \tag{18}$$

or by using the displacement functions

$$p_o\pi + q_o = k_o\int_{-\theta_o}^{\theta_o}\left[\overline{T}_o(\tau)+\overline{T}_1(\tau)\cos\theta+\sum_{n=0}^{\infty}T_n(\tau)w_n(\tau)\cos(n\theta)\right]d\theta + \int_0^1\int_0^{2\pi}\left[\ddot{\overline{T}}_o(\tau)+\ddot{\overline{T}}_1(\tau)r\cos\theta+\sum_{n=0}^{\infty}\ddot{T}_n(\tau)w_n(r)\cos(n\theta)\right]rdrd\theta \tag{19}$$

Solution of the governing Eq. (15) yields the static and the dynamic behavior of the plate, including free and forced vibrations. The free and forced vibration shapes of the completely free plate due to the initial conditions and due to the time dependent loading comes into being as a combination of the free vibrations of the plate including the rigid translation of the plate in the vertical direction and the rotation of the plate along its diameter and the elastic vibration mode shapes of the completely free plate, which can be seen in Eq. (7). When the stiffness of the support is soft relative to that of the plate, the contribution of the rigid translation and rotation become pronounced in the displacement configuration. Consequently, the numerical accuracy can be achieved by taking into account very few elastic mode shapes. On the other hand, when the support gets stiffer, the elastic deformations become noticeable and sufficient accuracy can be obtained by considering more elastic mode shapes. Separate investigation of these free vibrations yields the corresponding vibration periods of the plate as, $\pi\sqrt{2/k_o}$ (translational vibration of rigid plate on an elastic edge support), $\pi\sqrt{1/k_o}$ (rotational vibration of rigid plate an elastic on edge support) and $2\pi/\lambda_n^2$ (vibrations of elastic plate on rigid edge support), respectively. It is worth to point out that care has been taken in using non-dimensional parameters, in geometry, loading and time to accommodate the overall numerical results as general as possible.

3. Numerical results

Numerical analysis is carried out by using MATLAB (2012b) software programming language and numeric computing environment for the static and dynamic loading cases. Effects of the various parameters of the system on the behavior of the plate are studied and the results are displayed in the figures, comparatively. A comprehensive treatment of the free vibration of the completely free circular plate are given by Leissa (1969) and the properties of the Bessel function and their derivatives by McLachlan (1955). The system (11) has infinite number of roots for A_n and λ_n. It is plausible to expect that the lift-off of the plate can come into being when the displacement shape is not axially symmetric. Therefore, the displacement function approximated by considering only the first solution of λ_n for a given n. In other words, in the numerical analysis the first mode shape in the radial direction is considered only due to the properties of the loading, whereas several mode shapes in the angular direction are considered and compared with a view to achieve an acceptable accuracy in the numerical results. Since the numerical results are given graphically, the level of accuracy of the numerical approach is examined accordingly, and it was seen that four roots of the system are satisfactorily accurate for this purpose. They are obtained for $\nu = 0.25$ and $n = 0,1,2,3$ as

$$\lambda_n = 2.9816;\ 4.5177;\ 5.9405;\ 7.2906$$

$$A_n = -8.89569\times10^{-2};\ -1.95760\times10^{-2};\ -5.60094\times10^{-3};\ -1.83914\times10^{-3} \tag{20}$$

In the computing process, the numerical analysis starts

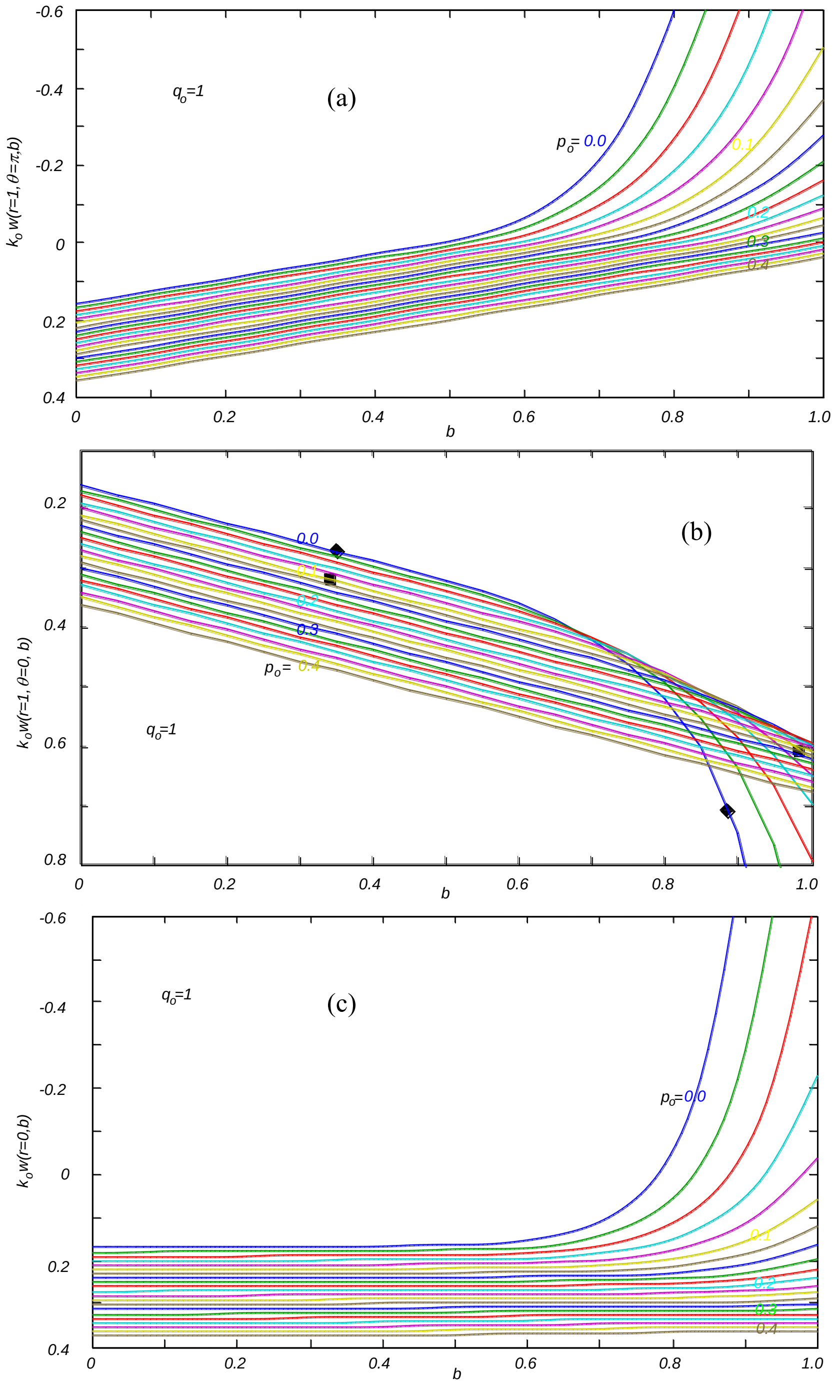

Fig. 3 (a) Displacement of the top edge point $w(r=1, \theta=\pi, b)$, (b) the bottom edge point $w(r=1, \theta=0, b)$ and (c) the middle point $w(r=0, b)$ for various values of the distributed load p_o for constant concentrated load q_o

with the solution of the nonlinear system of Eq. (11) for the static loading case by assuming a contact region on the edge of the plate, i.e., an assumption for $\pi \geq \theta_o \geq 0$ and continues with the evaluation of the integrals in the elements of the stiffness matrix. Solution of the nonlinear algebraic equation $\boldsymbol{K}(\theta_o)\boldsymbol{T}(\theta_o) = \boldsymbol{P}(\theta_o)$ is accomplished and the displacement function $w(r, \theta_o)$ is obtained. The contact angle θ_o is checked whether $w(1, \theta_o) = 0$ is satisfied and it is updated until a satisfactory result is found. In the dynamic case, additionally the mass matrix is evaluated, the initial displacement $w(r, \theta_o, \tau = 0)$ and the initial contact angle $\theta_o(\tau = 0)$ is found for the initial loading case, as it is in the static loading. Then, the differential Eq. (15) is solved in the time domain by updating the elements of the mass and the stiffness matrices at each time step by taking into account the variation of the contact angle $\theta_o(\tau)$ by adopting the linear acceleration procedure. After the displacement function is obtained, all parameters of the plate including the support reaction can be found. As it is known, the case $\theta_o = \pi$ corresponds to the complete contact along the edge of the plate to the unilateral springs and $\theta_o = 0$ corresponds to the complete

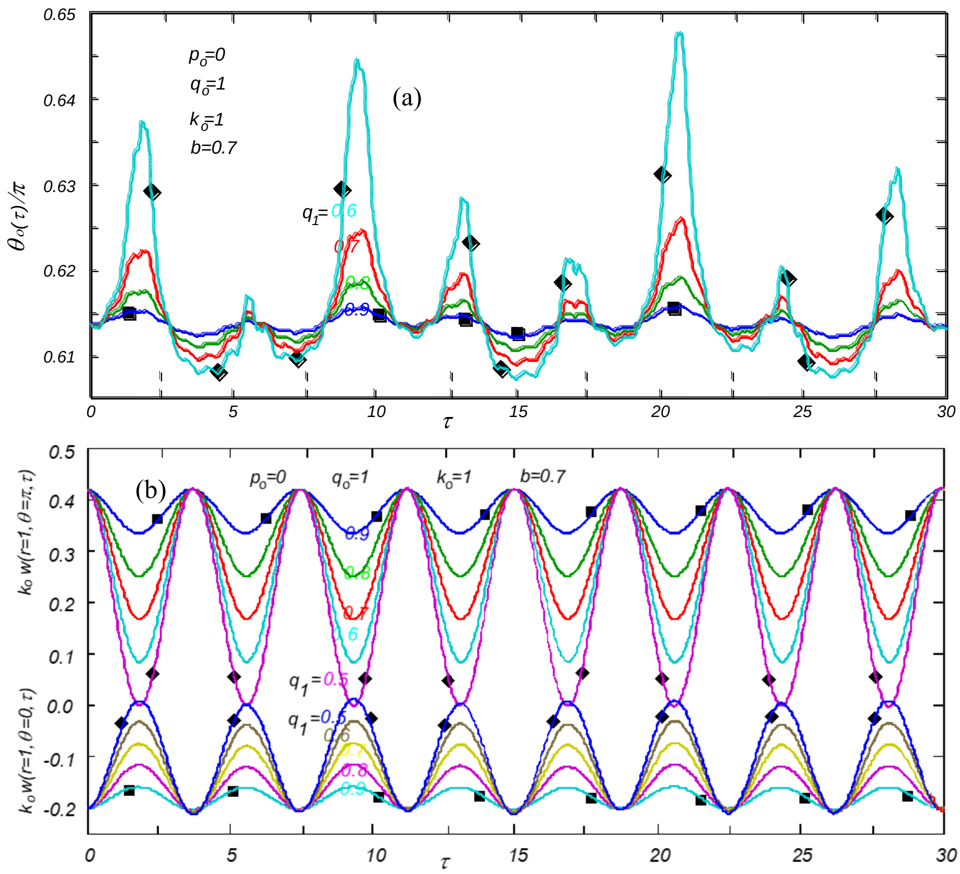

Fig. 4 (a) Contact angle $\theta_o(\tau)$ and (b) the top edge point $w(r=1,\theta=\pi,\tau)$ and the bottom edge point $w(r=1,\theta=\pi,\tau)$ of the plate due to the partial unloading from q_o to q_1

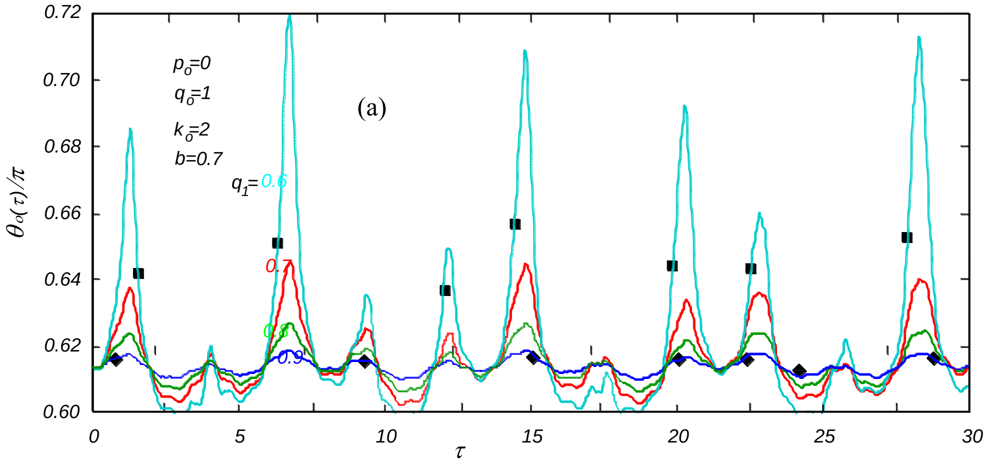

Fig. 5 (a) Contact angle $\theta_o(\tau)$ and (b) the top edge point $w(r=1,\theta=\pi,\tau)$ and the bottom edge point $w(r=1,\theta=\pi,\tau)$ of the plate due to the partial unloading from q_o to q_1 for the support stiffness $k_o=2$

separation of the plate from the support which may occur in the dynamic case.

Fig. 2 shows the contact angle $\theta_o(p_o/q_o,b)$ as a function of the loads q_o and p_o, and the eccentricity b, $b=0$ and $b=1$ corresponding to the cases q_o being applied at the center and at the edge of the plate, respectively. As the figure shows, the contact angle does not depend on the level of the loads, but on the ratio of the loading q_o/p_o. An increase in the loading level only enlarges the displacements without changing the extent of the contact zone. Furthermore, it does not depend on the spring constant at the edge, as it is reported in various other studies (Kamiya 1977, Celep, Turhan and Al-Zaid 1998a, 1988b, Celep 1988, Hong *et al.* 1999, Celep and Turhan 1990), whereas the support reaction depends on the stiffness of the support as $k_o w(r=1,\theta,p_o,q_o,b)H(\theta)$. When only the concentrated force q_o exists ($p_o=0$) at the edge the contact is established only at the point under the force. Besides,

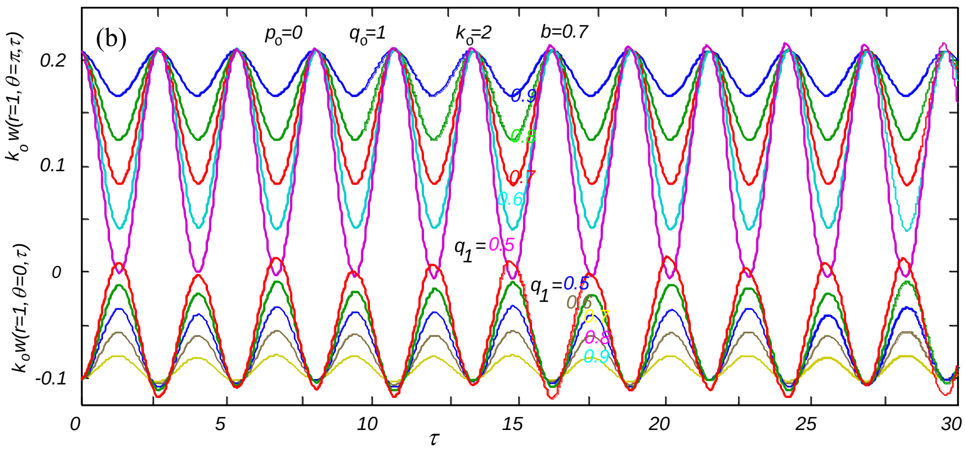

Fig. 5 Continued

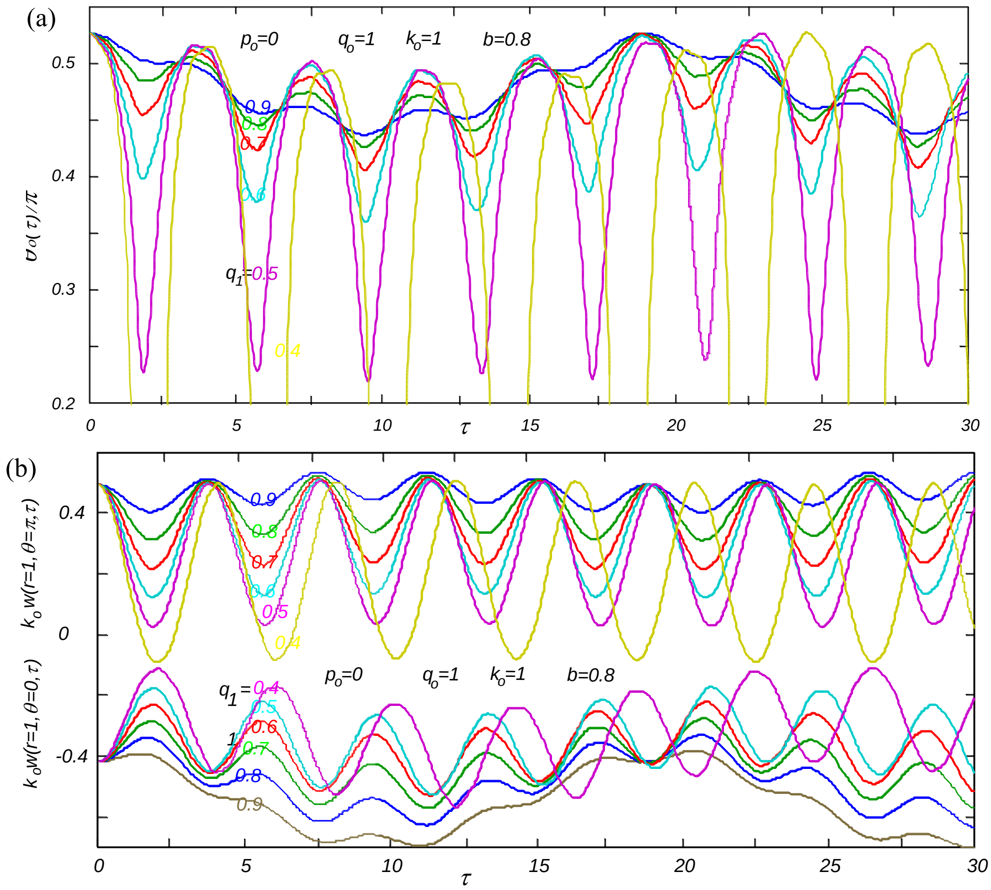

Fig. 6 (a) Contact angle $\theta_o(\tau)$ and (b) the top edge point $w(r=1, \theta=\pi, \tau)$ and the bottom edge point $w(r=1, \theta=\pi, \tau)$ of the plate due to the partial unloading from q_o to q_1 for the eccentricity of the concentrated load $b=0.8$

when the eccentricity of the load decreases the contact extent increases and the full contact develops. Furthermore, the figure shows the variation of the contact angle obtained in the present and that obtained by using SAP2000 software and in the study by Celep *et al.* (1988b) being in close agreement with each other. Similar variations for the contact angle as well as for the displacements can be found in the study by Celep and Gençoğlu (2003) for the rigid circular plate. Fig. 3 (a), (b) and (c) illustrate the displacements at the top, the bottom and the middle points of the plate, i.e., $w(r=1, \theta=\pi)$, $w(r=1, \theta=0)$ and $w(r=0)$, respectively, for various values of the distributed load p_o as a function of the eccentricity b by assuming a constant concentrated load q_o. As seen, for the small, distributed load p_o and for large eccentricity b, the variation displays nonlinear behavior due to the lift-off of the upper plate edge from the support. As expected, the problem is linear, when the complete contacts is established ($\theta_o(t) = \pi$). The linear variations in the figure indicate that nonlinearity does not appear, i.e., the complete contact is

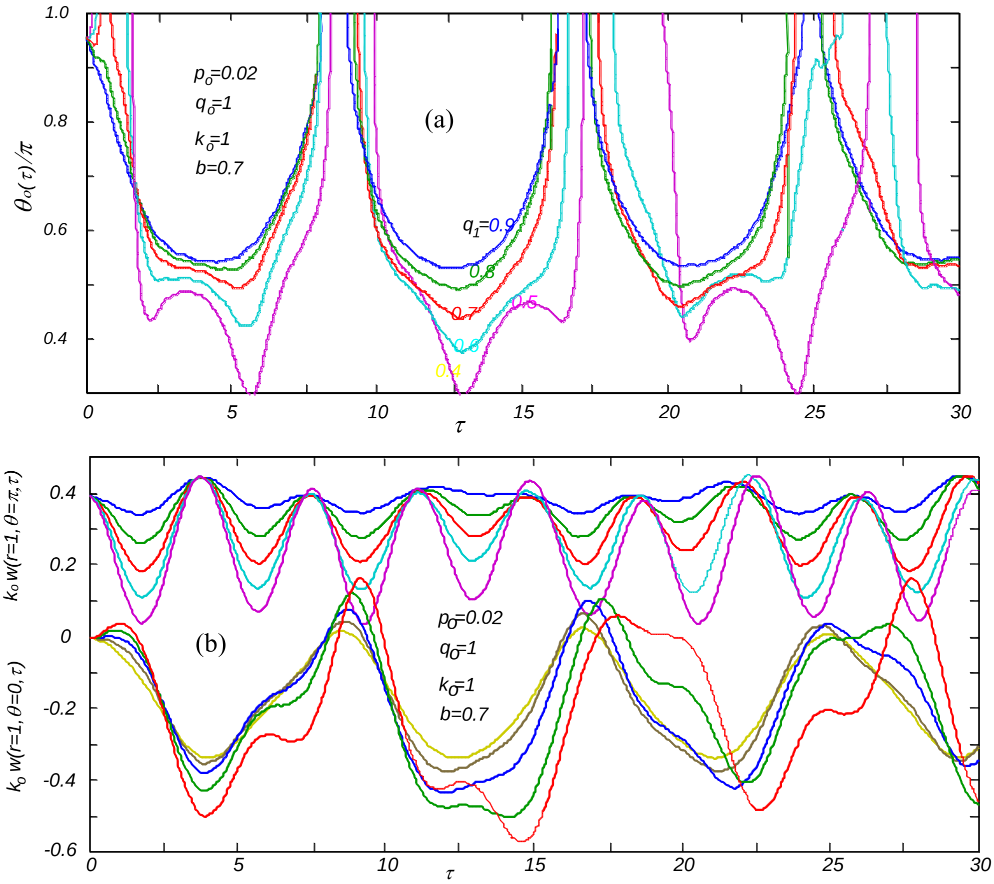

Fig. 7 (a) Contact angle $\theta_o(\tau)$ and (b) the top edge point $w(r=1,\theta=\pi,\tau)$ and the bottom edge point $w(r=1,\theta=\pi,\tau)$ of the plate due to the partial unloading from q_o to q_1 for the uniformly distributed load $p_o = 0.02$

established. For the dynamic solution, the plate is assumed to be in equilibrium initially under the prescribed loads p_o and q_o, however, the load q_o is decreased to q_1 at $\tau = 0$, while p_o is kept constant. Time variations of the contact angle $\theta_o(t)$, the top and the bottom edge points of the plate $w(r=1,\theta=\pi,\tau)$ and $w(r=1,\theta=0,\tau)$ due to this partial unloading of the system are evaluated and presented in Fig. 4, respectively. The results for a specific loading case $p_o = 0$, $q_o = 1$, $b = 0.7$ and the support stiffness $k_o = 1$ are selected for illustrative purposes. In fact, the parameters are chosen so that the nonlinearity appears to be noticeable. In fact, Fig. 1 shows the nonlinearity is effective at the beginning as an initial condition of the motion due to the partial contact and nonlinearity persists, when partial contact remains ($0 < \theta_o(t) < \pi$). As seen, the nonlinearity is much more pronounced in the variation of $\theta_o(t)$, whereas the nonlinearity is less noticeable in the variations of the displacements. The results related to the support stiffness $k_o = 2$, to the eccentricity of the concentrated load $b = 0.8$ nd to the uniformly distributed load $p_o = 0.02$ are presented in Fig. 5, Fig. 6 and Fig. 7, respectively, to illustrate their effect on the behavior of the plate. It deserves noting that the time step in the time domain analysis is selected to be less than 1/10 of the smallest of the vibration period of the highest mode shape to provide acceptable accuracy. However, it is recommended to compare the numerical integration results using different time steps to find the effect of accumulation of errors in time integrations that are constrained for linear systems but not guaranteed to be constrained for nonlinear systems.

4. Conclusions

Evaluation of static and dynamic deflections of circular plates is of great importance, as it is related with various structural and geotechnical engineering structures. The paper presents an investigation on the static and dynamic behavior of an elastic circular plate supported along its edge by unilateral springs, i.e., springs that react only to compression, but they cannot withstand any tension reaction, where lift-off develops. Although the analysis is carried out by considering a uniformly distributed load p_o and a concentrated load q_o and numerical results are presented for non-dimensional parameters of the plate and the loading, the analysis can be extended very easily for any type of loading. However, the results of the complicated load cases cannot be easily presented graphically and interpreted due to various possible lift-off and contact regions. In case of the lift-off, the problem is dealt numerically by using an iterative process in the case of static loading. When a partial contact develops, the displacements in the lift-off region get larger and the reaction in the contact region increases substantially to maintain vertical force equilibrium and causes a marked difference between the displacements of the tensionless and the conventional foundation models. In case of the dynamic loading, the contact region is updated at each time step. Since the elements of the stiffness matrix depend on the contact angle, they need to be updated by evaluating the integrals accordingly. Satisfaction of the global vertical force equilibrium in the plate by including the inertia forces, the external loads, and the support reaction at the edge of the plate are also checked, for a general control of the numerical solution. Overall, the theoretical formulation of the static and dynamic behavior of the circular plate on unilateral edge load is established using the Lagrange equations of motion by combining with the classical potential energy method. Although the formulation given for the distributed load and an eccentric concentrated load, the analysis can be extended easily to different type of loading by paying attention to various possible lift-off and contact regions. Specific conclusions of the paper can be summarized as follows

• When the circular plate is subjected to one of the static loading p_o or q_o, the contact region along the circular edge $-\theta_o \leq \theta \leq \theta_o$ does not depend on the level of loading. However, when the plate subjected to the two types of loading, the extent of the contact edge region depends on the ratio of the loading q_o/p_o.

• Examination of the numerical results shows that the nonlinear behavior of the problem becomes more marked in the time variation of the contact angle, whereas the nonlinearity is less noticeable in the time variation of the displacements.

• Since the unilateral springs do not develop any tensile forces, the time variation of the support reactions are less constrained and the displacements are larger compared to those of the conventional foundation. Consequently, if this result can be generalized as a strongly marked difference appears in general between the displacement configurations of the structural elements supported by the unilateral and the conventional edge supports.

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